Z-SCORES or STANDARD SCORES

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1. Why bother with Z scores?

For at least two main reasons.

Firstly, they play a central role in applied psychometrics.

Secondly, they make life much easier.

For example if you want to predict a score on Test Y from a score on Test X, you can either use the Z score formula or the raw scare alternative,

The raw score alternative is:

$$\hat{Y} = r_{xy} \frac{\sigma_y}{\sigma_x} (X - M_x) + M_y$$

In Z scores the formula becomes:

$$\hat{Z}y = r_{xy} Zx$$

Which would you prefer to have to use?

2. What are Z scores?

Z scores are scores which have been converted into the number of standard deviations that the test score being converted is above or below the mean. So a Z score of 1.5 means that the test score is 1.5 standard deviations above the mean.

This being so it is not surprising that the formula for converting a test score into a Z score is (the test score minus the mean) / the standard deviation

$$Z = \frac{X - M_x}{\sigma_x} \qquad 2.1$$

Test yourself

Work out the Z score corresponding to each of the following test scores on a test with a mean of 20 and a standard deviation of 5.

a. 10
b. 25
c. 5
d. 30

Answers

a. -2; b. +1; c. -3; d. +2

NB The direction of the difference from the mean is important so make sure the plusses and minuses are there

3. The mean of Z scores

The mean Z score is zero. This is not an arbitrary assignment of a number. It follows from the definition of a Z score

The formula for the mean Z score will of course be:

$$Mz = \frac{\Sigma Z}{N} \qquad 2.2$$

which in terms of test scores will be

$$\frac{\Sigma(X - Mx)}{N\sigma_x} \qquad 2.3$$

But the upper part of this formula equals the sum of deviation scores from the mean. This sum always equals zero. This is because

$$\sum (X - Mx) = \sum X - NMx \qquad 2.4$$

(there are 'N' X scores and each has the mean subtracted from it, so, in all, N means are subtracted)

But the formula for the mean is

$$\sum X/N = Mx. \qquad 2.5$$

If we multiply both sides of this equation by N we find that

$$\sum X = NMx. \qquad 2.6$$

Thus $\sum X - NMx$ must equal zero

Returning to the formula for the mean of Z scores

$$\frac{\Sigma(X - Mx)}{N\sigma_x} \qquad 2.7$$

It can be seen that the numerator is $\sum (X - Mx)$ which is equal to zero. Therefore the mean of Z scores is zero.

4. The variance of Z scores

The variance of Z scores will be 1, because:

$$\sigma_z^2 = \frac{\sum\left(\frac{X - M_x}{\sigma_x}\right)^2}{N} = \frac{\Sigma(X - M_x)^2}{N\sigma_x^2} \qquad 2.8$$

But

$$\sigma_x^2 = \frac{\Sigma(X - M_x)^2}{N} \qquad 2.9$$

So

$$\Sigma(X - Mx)^2 = N\sigma_x^2 \qquad 2.10$$

So (2.8) becomes

$$\sigma_z^2 = \frac{N\sigma_x^2}{N\sigma_x^2} = 1.0 \qquad 2.11$$

The variance of Z scores = 1. The standard deviation is the square root of the variance . The square root of 1 is 1, so the standard deviation of Z scores is 1.

Thus Z scores will have a mean of zero and a standard deviation of 1.

5. Using Z scores

The formula for the normal distribution is

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(X-M)^2/2\sigma^2} \qquad 2.12$$

the only reason for presenting this formula here is so that you can see that it contains both the mean and standard deviation amongst it component parts.

Tables for the standard normal distribution are based on score with a mean of zero and a standard deviation of 1. Thus, once we have a Z score we can use tables for the standard normal distribution to find out what proportion of cases would fall below or fall above any particular Z score.

Here is an example. This table shows the proportion of cases falling below a given Z.

Z	Cumulative proportion	Z	Cumulative proportion	Z	Cumulative proportion
-3.5	0.0002	-1.1	0.1357	1.3	0.9032
-3.4	0.0003	-1	0.1587	1.4	0.9192
-3.3	0.0005	-0.9	0.1841	1.5	0.9332
-3.2	0.0007	-0.8	0.2119	1.6	0.9452
-3.1	0.0010	-0.7	0.2420	1.7	0.9554
-3	0.0013	-0.6	0.2743	1.8	0.9641
-2.9	0.0019	-0.5	0.3085	1.9	0.9713
-2.8	0.0026	-0.4	0.3446	2	0.9772
-2.7	0.0035	-0.3	0.3821	2.1	0.9821
-2.6	0.0047	-0.2	0.4207	2.2	0.9861
-2.5	0.0062	-0.1	0.4602	2.3	0.9893
-2.4	0.0082	0	0.5000	2.4	0.9918
-2.3	0.0107	0.1	0.5398	2.5	0.9938
-2.2	0.0139	0.2	0.5793	2.6	0.9953
-2.1	0.0179	0.3	0.6179	2.7	0.9965
-2	0.0228	0.4	0.6554	2.8	0.9974
-1.9	0.0287	0.5	0.6915	2.9	0.9981
-1.8	0.0359	0.6	0.7257	3	0.9987
-1.7	0.0446	0.7	0.7580	3.1	0.9990
-1.6	0.0548	0.8	0.7881	3.2	0.9993
-1.5	0.0668	0.9	0.8159	3.3	0.9995
-1.4	0.0808	1	0.8413	3.4	0.9997
-1.3	0.0968	1.1	0.8643	3.5	0.9998
-1.2	0.1151	1.2	0.8849		

Test yourself

What proportion of cases will fall; (a) below a Z of –2; (b) below a Z of +1; (c) below a Z of +.7; (d) below a Z of –1?

Answers

(a) .0228; (b) .8413 (c) .7580; (d) .1587

You can also see what the answers should be by using the Z lookup calculator available at www.psychassessment.com.au

6. Converting scores from one scale to another

Test scales and norms differ in terms of their means and standard deviations.

The table below provides data for some commonly used clinical tests

Test score	Mean	Standard Deviation
Wechsler IQs	100	15
WAIS III factor scores	100	15
Wechsler subtests	10	3
Cattell Sten Scores	5.5	2
T scores	50	10

Suppose that somebody obtains the following test results:

WAIS Full Scale IQ = 120; WAIS Information subtest 8; Cattell 16 PF Factor B (intelligence) sten 8; and a T score of 40 on a newly developed test of General Knowledge. How comparable are these scores? It looks as though there are differences between them, but how big are those differences?

Obviously we need to convert the results to a common scale. And we can do this by converting each result into a Z score.

The Zs are shown below

WAIS IQ = (120 – 100)/15 = 1.33

WAIS Information – (8 - 10)/3 = -.67

Cattell B = (8 - 5.5)/2 = 1.25

General Knowledge Test = (40 – 50)/10 = - 1.0

The catch here is that lots of people are unfamiliar with Z scores. It might be better to convert the scores into Wechsler IQ equivalents, ie, give all scales a mean of 100 and a standard deviation of 15.

This is easily done once we have the Z scores, because the formula for converting a Z to any scale you like is simply

Score on new scale = Z times the standard deviation of the new scale plus the mean of the new scale.

The derivations of this formula and its raw score equivalent are shown below

$$\frac{Y - M_y}{\sigma_y} = \frac{X - M_x}{\sigma_x}$$

$$Y - M_y = \frac{X - M_x}{\sigma_x} \bullet \sigma_y$$

$$Y = \frac{X - M_x}{\sigma_x} \bullet \sigma_y + M_y = Z_x \sigma_y + M_y$$

or

$$Y = \frac{\sigma_y}{\sigma_x}(X - M_x) + M_y \qquad 2.13$$

So, if we apply this formula to all the scores above we get the following results

WAIS IQ = **120**

WAIS Information = (- .67 x 15) + 100 = **90**

Cattell B = (1.25 x 15) + 100 = **119**

General Knowledge = (-1 x 15) + 100 = **85**

The question of whether or not the observed differences are either reliably or abnormally large will be dealt with in Module X.

Test yourself

Convert the scores above into T scores (Mean 50, standard deviation 10)

Answers

WAIS 120 = T 63.33

WAIS Information 8 = T 43.3

Cattell B 8 = T 62.5

General Knowledge = T 40

One type of score is conspicuous by its absence in the example above. None of the scores was reported as a percentile. And percentiles do not have a meaningful mean and standard deviation.

So supposing that one of the tests had been another intelligence test. Let's say it was Raven's Progressive Matrices, and that on this test the score was at the 90th percentile.

How do we convert this to an equivalent Wechsler IQ?

We do it by reference to tables relating percentiles to Z scores.

And below is a table for doing just that. Consulting the Table shows that the 90th percentile corresponds to a Z of 1.28. So the score translated into Wechsler IQ units will be: (1.28 x 15) + 100 = 119

Table for converting Percentiles to Zs

Percentile	Z	Percentile	Z	Percentile	Z
99	2.33	66	0.41	33	-0.44
98	2.05	65	0.39	32	-0.47
97	1.88	64	0.36	31	-0.50
96	1.75	63	0.33	30	-0.52
95	1.64	62	0.31	29	-0.55
94	1.55	61	0.28	28	-0.58
93	1.48	60	0.25	27	-0.61
92	1.41	59	0.23	26	-0.64
91	1.34	58	0.20	25	-0.67
90	1.28	57	0.18	24	-0.71
89	1.23	56	0.15	23	-0.74
88	1.17	55	0.13	22	-0.77
87	1.13	54	0.10	21	-0.81
86	1.08	53	0.08	20	-0.84
85	1.04	52	0.05	19	-0.88
84	0.99	51	0.03	18	-0.92
83	0.95	50	0.00	17	-0.95
82	0.92	49	-0.03	16	-0.99
81	0.88	48	-0.05	15	-1.04
80	0.84	47	-0.08	14	-1.08
79	0.81	46	-0.10	13	-1.13
78	0.77	45	-0.13	12	-1.17
77	0.74	44	-0.15	11	-1.23
76	0.71	43	-0.18	10	-1.28
75	0.67	42	-0.20	9	-1.34
74	0.64	41	-0.23	8	-1.41
73	0.61	40	-0.25	7	-1.48
72	0.58	39	-0.28	6	-1.55
71	0.55	38	-0.31	5	-1.64
70	0.52	37	-0.33	4	-1.75
69	0.50	36	-0.36	3	-1.88
68	0.47	35	-0.39	2	-2.05
67	0.44	34	-0.41	1	-2.33

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7. Normalising distributions of scores via percentiles and Z scores

7. 1. Calculating percentiles

Percentiles can be calculated form individual or grouped scores. If the scores are grouped the number of categories should be as large as is reasonably possible, and the number of sscores should be reasonably large.

Having laid down these two rules they will be largely ignored in the following example. BUT they have only been ignored to make the sums and the procedure clearer.

Suppose we have the test scores shown in the in the first two columns of the table below.

The first thing we need to do in converting the scores to percentiles is to work out the cumulative frequency (the number scoring at or below a given score). This is shown in the third column.

The next column gives the cumulative frequency BELOW a given score plus half the number of cases in the given score category. Why is this done? Because it is assumed that those obtaining a given score are spread evenly through it, so, taking the portion of the score which separates those in the top half of its range from those in the bottom half seems a reasonable idea when we want to assign a percentile to that score. It is an arbitrary but reasonable procedure.
Finally, in the last column, we have the percentile corresponding to a given score.

It is unlikely that you will ever need to use this procedure. But there could be occasions when you are faced with obviously highly skewed norms for a diagnostic test and want to give a finer discrimination than simply stating that the patient is above or below the cut-off. Fortunately most test developers will have done this work for you.

Calculating percentiles				
Score	Frequency of score (*f*)	Cumulative frequency of score (*cf*)	Cumulative frequency below (*cfb*) plus .5 *f*	Percentile *((cfb + .5f)/N) x 100*
50	2	20	19	95
49	3	18	16.5	82.5
48	4	15	13	65
47	5	11	8.5	42.5
46	3	6	4.5	22.5
45	2	3	2	10
44	1	1	.5	2.5

IMPORTANT NOTE ABOUT CHANGES NOTED IN PERCENTILES ON RE-TESTING

Percentiles have a peculiarity which it is worth mentioning here. They are tricky things to use if you are trying to assess changes in score.

For example suppose somebody's IQ on a Wechler type test goes up by 10 percentile points. How much change in IQ units does this represent?

There is no fixed answer. It depends on what the starting percentile was.

For example, if someone's IQ goes up from the fifth to the 15th percentile, the change in IQ will be approximately 14 IQ points.

On the other hand a rise in percentile from 45th to 55th – again a rise of 10 percentile points – will equate to only 4 IQ points.

So a change in percentiles of a given size will be associated with quite different amounts of change in the underlying scale.

Test yourself

Suppose the rises were from (a) the 5th to the 25th percentile and (b) from the 40th percentile to the 60th, what would the corresponding rises in IQ be?

Not sure how to work it out? Do it this way.

Step 1 find the Z scores corresponding to each of the percentile mentioned – 5th, 25th, 40th and 60th.

Step 2. Subtract the 5th %ile Z from the 25th%ile Z

Step 3. Multiply the difference found by the standard deviation of the test (in this case 15). The answer is the number of IQ points corresponding to the change in percentiles.

Step 4. Repeat for the 40th and 60th %iles.

Answers

(a) 19 IQ points;

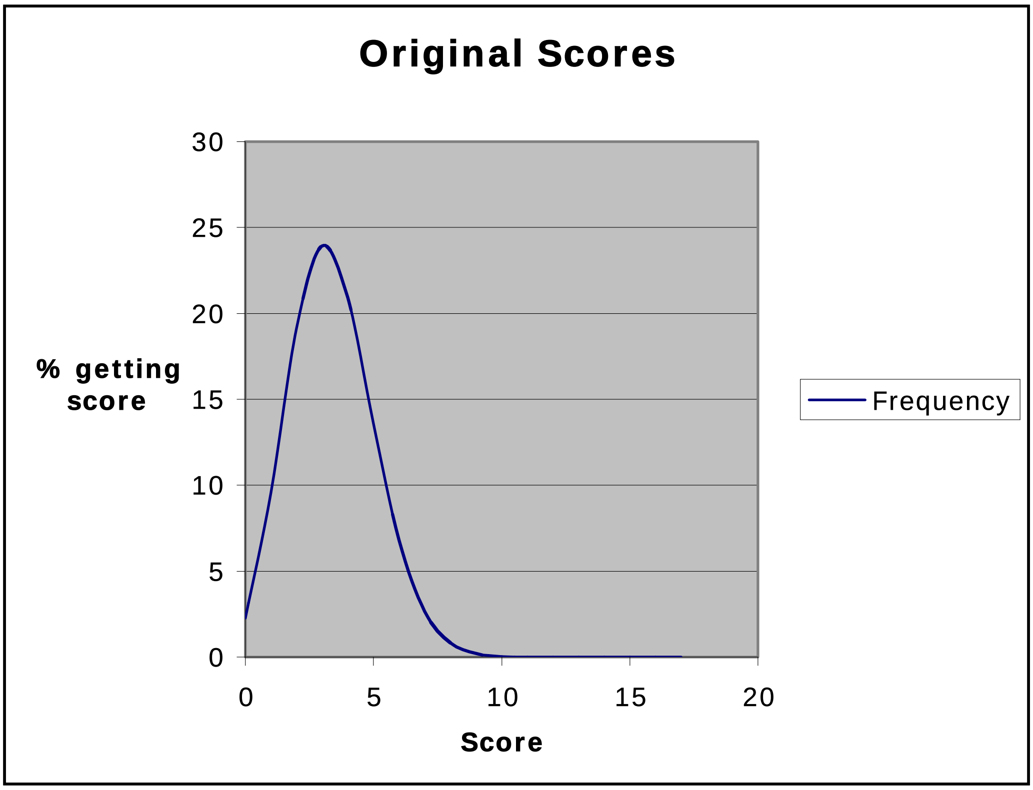
(b) 8 IQ points

7.2. 'Normalising' via T-Scaling

Suppose that we are constructing an intelligence test and we want one of the subtests to be a vocabulary test. And we would like this test to have a mean of 10 and a standard deviation of 3 – just like Wechsler subtest.

When we administer the subtest to our standardisation group we get the distribution of scores shown below.

As you can see the distribution is positively skewed.



But we want a subtest with normally distributed scores, a mean of 10 and a standard deviation of 3.

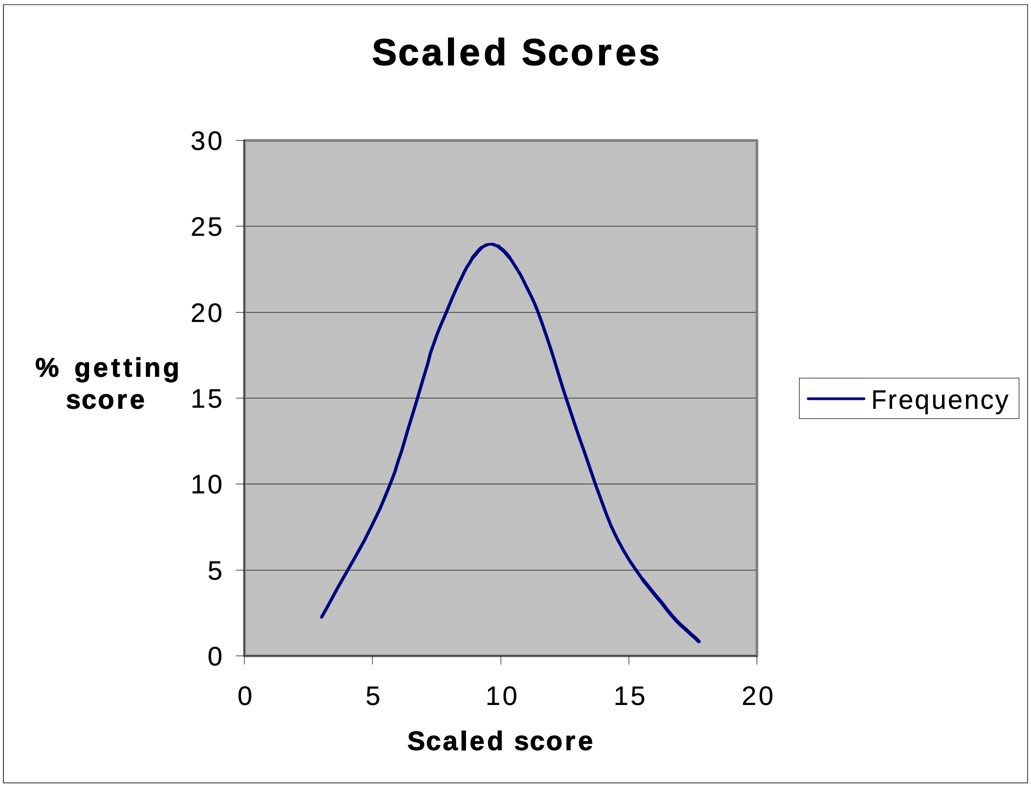
How do we get one?

Step 1. Work out the percentiles corresponding to each of the original scores by the method already described.

Step 2. Convert the percentiles into the corresponding Z score on the standard normal distribution.

Step 3. Multiply the Z score so obtained by the desired standard deviation (in this case 3) and add to the answer the desired men (in this case 10)

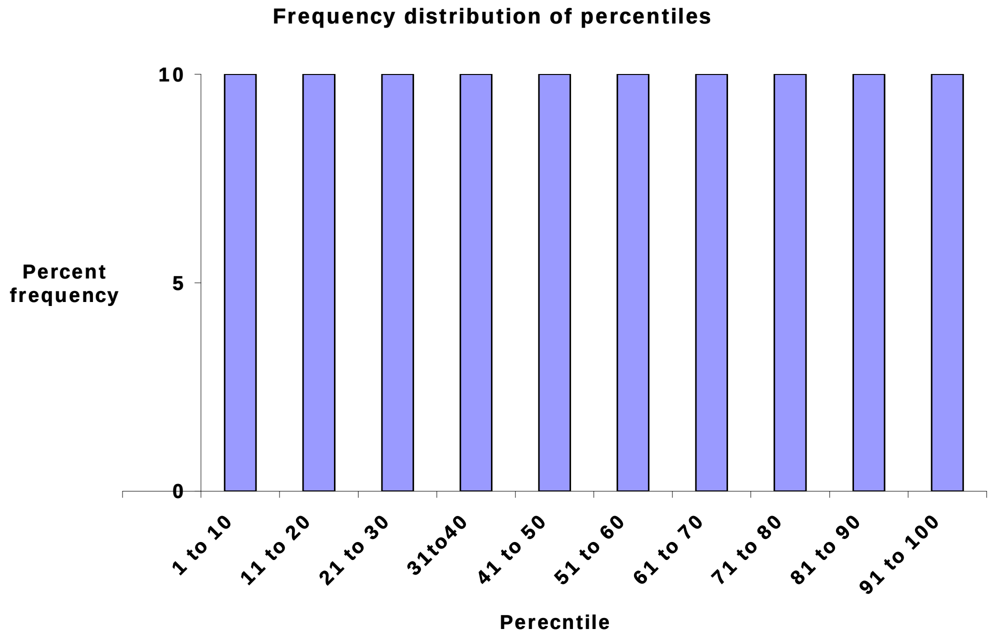
As a result of this procedure the distribution above is transformed into the distribution below.



This method is the one most commonly used for normalising test scores and ensuring that test and subtest scores are normally distributed with a given mean and standard deviation.

BUT note that normalising was achieved by going from score to percentile to Z score. **Simply converting score to Z does NOT normalise the distribution.**

And simply converting scores to percentiles will give a rectangular distribution like this:



8. Some important values of Z.

The table below shows the Zs corresponding to key probabilities. These values will be very useful when we come to discuss the reliability and abnormality of differences. The table will be available on request via a button or icon.

Sometimes an abnormal' score is defined in terms of rarity. So it might be decided that scores occurring with a population frequency of five percent or less should be considered abnormal. This would mean that on, say, a Wechsler Block Design subtest. A score 1.65 standard deviations above or below the man should be considered abnormal. At least, abnormal in the sense of raising the question of the need for possible further inquiry
These values (using the table below) will be 10 plus or minus 1.65 x 3. This equals values below 5.05 or above 14.95

Important values of Z		
Probability/proportion	Z	Z
P	One Tail	Two Tail
.05	1.65	1.96
.025	1.96	2.24
.01	2.33	2.58
.005	2.58	2.81
.001	3.08	3.31

Test yourself

What would the values be if the criterion chosen had been scores occurring with a population frequency of one percent or less?

Answer: below 3.01 or above 16.99

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9. A final thought - what is a percentile anyway?

For more insight into the answer to this question, read Professor Paul Barrett's paper on this topic, View Paper Here

Or visit: www.pbarrett.net